Unit 5
Continuity & Differentiability

SECTION – A

Questions 1 to 10 carry 1 mark each.

1. The function f(x) = [x], where [x] denotes the greatest integer function, is continuous at:
(a) 4 (b) 1.5 (c) 1 (d) – 2
Ans. (b) 1.5

2. If $x = at^2$, $y = 2at$, then $\frac{d^2y}{dx^2}$ is

(a) $\frac{1}{t}$ (b) $-\frac{1}{t^2}$ (c) at^2 (d) $-\frac{1}{2at^3}$

Ans: (d) $-\frac{1}{2at^3}$

3. If $y = \sin^{-1}\left(\frac{3x}{2} - \frac{x^3}{2}\right)$, then $\frac{dy}{dx}$ is

(a) $\frac{3}{\sqrt{4-x^2}}$ (b) $\frac{-3}{\sqrt{4-x^2}}$ (c) $\frac{1}{\sqrt{4-x^2}}$ (d) $\frac{-1}{\sqrt{4-x^2}}$

Ans: (a) $\frac{3}{\sqrt{4-x^2}}$

(a), as $y = \sin^{-1}\left\{3 \cdot \frac{x}{2} - 4 \cdot \left(\frac{x}{2}\right)^3\right\} = 3\sin^{-1}\frac{x}{2}$

$$\therefore \frac{dy}{dx} = 3 \cdot \frac{1}{\sqrt{1-\frac{x^2}{4}}} \cdot \frac{1}{2} = \frac{3}{\sqrt{4-x^2}}$$

4. If $y = Ae^{5x} + Be^{-5x}$ then $\frac{d^2y}{dx^2}$ is equal to

(a) $25y$ (b) $5y$ (c) $-25y$ (d) $10y$
Ans: (a), as $y' = 5Ae^{5x} - 5Be^{-5x}$
and $y'' = 25Ae^{5x} + 25Be^{-5x} = 25y$

5. Derivative of $\sin x$ with respect to $\log x$, is

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(a) $\frac{x}{\cos x}$ (b) $\frac{\cos x}{x}$ (c) $x \cos x$ (d) $x^2 \cos x$

Ans: (c), let $y = \sin x$ and $t = \log x$,

$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt} = \cos x \times \frac{x}{1} = x \cos x$

6. The function f(x) = x|x| is
(a) continuous and differentiable at x = 0.
(b) continuous but not differentiable at x = 0.
(c) differentiable but not continuous at x = 0.
(d) neither differentiable nor continuous at x = 0.
Ans: (a) continuous and differentiable at x = 0.

7. A function $f(x) = \begin{cases} \frac{\sin x}{x} + \cos x, x \neq 0 \\ 2k \quad ,x = 0 \end{cases}$ is continuous at $x = 0$ for

(a) $k = 1$ (b) $k = 2$ (c) $k = \frac{1}{2}$ (d) $k = \frac{3}{2}$

Ans: (a), as $\lim_{x \to 0}\left(\frac{\sin x}{x} + \cos x\right) = 1 + 1 = 2 = 2k \Rightarrow k = 1$

8. If $y = \tan^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{1}{1+x^4}$ (b) $\frac{-2x}{1+x^4}$ (c) $\frac{-1}{1+x^4}$ (d) $\frac{x^2}{1+x^4}$

Ans: (b) $\frac{-2x}{1+x^4}$

(b), $y = \tan^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \tan^{-1}\left(\frac{\pi}{4}\right) - \tan^{-1} x^2$

$y' = 0 - \frac{1}{1+x^4} \cdot 2x = \frac{-2x}{1+x^4}$

In the following questions 9 and 10, a statement of assertion (A) is followed by a statement of reason (R). Mark the correct choice as:
(a)Both assertion (A) and reason (R) are true and reason (R) is the correct explanation of assertion (A).
(b)Both assertion (A) and reason (R) are true but reason (R) is not the correct explanation of assertion (A).
(c)Assertion (A) is true but reason (R) is false.
(d)Assertion (A) is false but reason (R) is true.

9. **Assertion (A):** Every differentiable function is continuous but converse is not true.
Reason (R): Function f(x) = |x| is continuous.
Ans: (b) Both assertion (A) and reason (R) are true but reason (R) is not the correct explanation of assertion (A).

10. **Assertion (A):** f(x)= |x – 3| is continuous at x = 0.
Reason (R): f(x) = |x – 3| is differentiable at x = 0.
Ans: (b) Both assertion (A) and reason (R) are true but reason (R) is not the correct explanation of assertion (A).

SECTION – B

Questions 11 to 14 carry 2 marks each.

11. Find all points of discontinuity of f, where f is defined by $f(x)=\begin{cases}2x+3, & if\ \ x\le 2\\ 2x-3, & if\ \ x>2\end{cases}$.

Ans. Here, $f(x)=\begin{cases}2x+3, & if\ \ x\le 2\\ 2x-3, & if\ \ x>2\end{cases}$.

At x = 2, $LHL=\lim_{x\to 2^-} f(x)=\lim_{x\to 2^-}(2x+3)$

Putting $x = 2 - h$ has $x \to 2^-$ when $h\to 0$

$\therefore \lim_{x\to 2^-} f(x)=\lim_{h\to 0}(2(2-h)+3)=\lim_{h\to 0}(4-2h+3)=\lim_{h\to 0}(7-2h)=7$

At x = 2, $RHL=\lim_{x\to 2^+} f(x)=\lim_{x\to 2^+}(2x-3)$

Putting $x = 2 + h$ as $x\to 2^+$ when $h\to 0$

$\therefore \lim_{x\to 2^+} f(x)=\lim_{h\to 0}(2(2+h)-3)=\lim_{h\to 0}(4+2h-3)=\lim_{h\to 0}(1+2h)=1$

$\therefore$ LHL $\neq$ RHL. Thus, $f(x)$ is discontinuous at $x = 2$.

12. Find the values of k so that the function f $f(x)=\begin{cases}kx+1, & if\ \ x\le 5\\ 3x-5, & if\ \ x>5\end{cases}$ is continuous at point x = 5.

Ans: Here, $f(x)=\begin{cases}kx+1, & if\ \ x\le 5\\ 3x-5, & if\ \ x>5\end{cases}$

At x = 5, $LHL=\lim_{x\to 5^-} f(x)=\lim_{x\to 5^-}(kx+1)$

Putting $x = 5 - h$ has $x\to 5^-$ when $h\to 0$

$\therefore \lim_{x\to 5^-} f(x)=\lim_{h\to 0}(k(5-h)+1)=\lim_{h\to 0}(5k-kh+1)=5k+1$

At x = 5, $RHL=\lim_{x\to 5^+} f(x)=\lim_{x\to 5^+}(3x-5)$

Putting $x = 5 + h$ as $x\to 5^+$; $h\to 0$

$\therefore \lim_{x\to 5^+} f(x)=\lim_{h\to 0}(3(5+h)-5)=\lim_{h\to 0}(10+3h)=10$

Also, f(5) = 5k + 1

Since f(x) is continuous at x = 5, therefore LHL = RHL = f(5)

$\Rightarrow 5k+1=10 \Rightarrow 5k=9 \Rightarrow k=\frac{9}{5}$

13. Differentiate sin $(\tan^{-1} e^{-x})$ with respect to x.

Ans: Let y = sin $(\tan^{-1} e^{-x})$

Differentiating both sides w.r.t. x, we get

$\frac{dy}{dx}=\frac{d}{dx}\left[\sin(\tan^{-1}e^{-x})\right]=\cos(\tan^{-1}e^{-x})\frac{d}{dx}(\tan^{-1}e^{-x})$

$=\cos(\tan^{-1}e^{-x})\frac{1}{1+(e^{-x})^2}\frac{d}{dx}(e^{-x})$

$=\cos(\tan^{-1}e^{-x})\frac{1}{1+e^{-2x}}(-e^{-x})=-\frac{e^{-x}\cos(\tan^{-1}e^{-x})}{1+e^{-2x}}$

14. Find $\frac{dy}{dx}$ if $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), 0 < x < 1$

Ans: Let x = $\tan\theta \Rightarrow \theta = \tan^{-1} x$, then we have

$$y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) = \cos^{-1}\cos 2\theta = 2\theta$$

$$\Rightarrow y = 2\tan^{-1} x$$

$$\Rightarrow \frac{dy}{dx} = 2 \times \frac{1}{1+x^2} = \frac{2}{1+x^2}$$

SECTION – C

Questions 15 to 17 carry 3 marks each.

15. Find $\frac{dy}{dx}$ if $x = a\,(\theta + \sin\theta), y = a\,(1 - \cos\theta)$.

Ans: Given that $x = a\,(\theta + \sin\theta), y = a\,(1 - \cos\theta)$

Differentiating w.r.t. θ, we get

$$\frac{dx}{d\theta} = a(1+\cos\theta), \frac{dy}{d\theta} = a(\sin\theta)$$

$$\text{Therefore, } \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a(\sin\theta)}{a(1+\cos\theta)} = \frac{\sin\theta}{1+\cos\theta} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \tan\frac{\theta}{2}$$

16. Prove that the function f given by $f(x) = |x - 1|, x \in \mathbb{R}$ is not differentiable at $x = 1$.

Ans: Given, $f(x) = |x-1| = \begin{cases} x-1, & \text{if } x-1 \geq 0 \\ -(x-1), & \text{if } x-1 < 0 \end{cases}$

We have to check the differentiability at $x = 1$

Here, $f(1) = 1 - 1 = 0$

$$Lf'(1) = \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h} = \lim_{h \to 0} \frac{1-(1-h)-0)}{-h}$$

$$= \lim_{h \to 0} \frac{h}{-h} = -1$$

and

$$Rf'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{(1+h)-1-0}{h}$$

$$= \lim_{h \to 0} \frac{h}{h} = 1$$

$\therefore Lf'(1) \neq Rf'(1)$.

Hence, f(x) is not differentiable at x = 1

17. If $y = 3e^{2x} + 2e^{3x}$, prove that $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$

Ans: Given that $y = 3e^{2x} + 2e^{3x}$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 6e^{2x} + 6^{3x} = 6(e^{2x} + e^{3x})$$

Again, Differentiating both sides w.r.t. x, we get

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$$\frac{d^2y}{dx^2} = 6(2e^{2x} + 3e^{3x})$$

Now, $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 6(2e^{2x} + 3e^{3x}) - 5(6(e^{2x} + e^{3x})) + 6(3e^{2x} + 2e^{3x})$

$= 12e^{2x} + 18e^{3x} - 30e^{2x} - 30e^{3x} + 18e^{2x} + 12e^{3x} = 0$

SECTION – D

Questions 18 carry 5 marks.

18. Differentiate $(\log x)^x + x^{\log x}$ with respect to x.

Ans: Let $y = (\log x)^x + x^{\log x}$

Let $u = (\log x)^x$ and $v = x^{\log x}$ then we have $y = u + v$

Therefore, $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ -------------------- (1)

Now, $u = (\log x)^x$

Taking logarithm on both sides, we have $\log u = x\log(\log x)$.

Differentiating both sides w.r.t. x, we have

$$\frac{1}{u}\frac{du}{dx} = x\frac{d}{dx}\log(\log x) + \log(\log x)\frac{d}{dx}(x) = \frac{x}{\log x} \times \frac{1}{x} + \log(\log x)$$

$$\Rightarrow \frac{du}{dx} = u\left[\frac{1}{\log x} + \log(\log x)\right] = (\log x)^x\left[\frac{1}{\log x} + \log(\log x)\right] \text{ --------------------- (2)}$$

Again $v = x^{\log x}$

Taking logarithm on both sides, we have $\log v = (\log x)\log x = (\log x)^2$

Differentiating both sides w.r.t. x, we have

$$\frac{1}{v}\frac{dv}{dx} = 2\log x\frac{d}{dx}(\log x) = 2\log x \times \frac{1}{x}$$

$$\Rightarrow \frac{dv}{dx} = v\left[\frac{2\log x}{x}\right] = x^{\log x}\left[\frac{2\log x}{x}\right] \text{ --------------------- (3)}$$

From (1), (2) and (3)

$$\frac{dy}{dx} = (\log x)^x\left[\frac{1}{\log x} + \log(\log x)\right] + x^{\log x}\left[\frac{2\log x}{x}\right]$$

$$\Rightarrow \frac{dy}{dx} = (\log x)^{x-1}\left[1 + \log x \log(\log x)\right] + 2x^{\log x - 1}.\log x$$

SECTION – E (Case Study Based Questions)

Questions 19 to 20 carry 4 marks each.

19. A potter made a mud vessel, where the shape of the pot is based on $f(x) = |x - 3| + |x - 2|$, where $f(x)$ represents the height of the pot.

(a) When x > 4 What will be the height in terms of x? (1)

(b) What is $\frac{dy}{dx}$ at x = 3? (1)

(c) When the x value lies between (2, 3) then the function is (1)

(d) If the potter is trying to make a pot using the function f(x) = [x], will he get a pot or not? Why? (1)

Ans: (a) 2x – 5

(b) function is not differentiable

(c) 1

(d) No, because it is not continuous

20. Sumit has a doubt in the continuity and differentiability problem, but due to COVID-19 he is unable to meet with his teachers or friends. So he decided to ask his doubt with his friends Sunita and Vikram with the help of video call. Sunita said that the given function is continuous for all the real value of x while Vikram said that the function is continuous for all the real value of x except at x = 3.

The given function is $f(x) = \frac{x^2 - 9}{x - 3}$

Based on the above information, answer the following questions:

(a) Whose answer is correct? (1)

(b) Find the derivative of the given function with respect to x. (1)

(c) Find the value of f '(3). (1)

(d) Find the second differentiation of the given function with respect to x. (1)

Ans: (a) Vikram (b) 1 (c) 1 (d) 0